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Kinks and Cobordism

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This paper is concerned with the situation in which the topology of space (or space–time) changes to produce a new manifold that is cobordant with, but not necessarily of the same homotopy type as, the original manifold. The relevance to kink field theories is discussed. It is shown that whenever the concept of degree of mapping is applicable then the degree is conserved under the bordism relation. This has the consequence that certain (topological) fermions arising in general relativity are always conserved in number, even when changes in topology are permitted.

1. INTRODUCTION

The classification of base-point-preserving mappings $\varphi: X \to Y$ in which the manifold Y is topologically nontrivial is currently of interest in elementary particle field theory. The different homotopy classes $[X, Y]$ of base-point-preserving mappings φ are associated with different types of particlelike structures called "kinks" (Finkelstein, 1966). For most theories of physical interest, the domain manifold X is taken to be S^3, or equivalently R^3 with φ mapping the region at infinity into some fixed point $y_0 \in Y$. General relativity, for which X may be chosen to be a more complicated manifold than S^3 or R^3, provides an important exception. For example, $X = S^1 \times S^2 \times R^1$ represents a space–time manifold with the "wormhole" topology.

The essential power of kink theory lies in two results: (i) the number of kinks is a conserved quantity; (ii) kinks can be used to describe fermions in terms of the more basic mesons. The conservation of kinks holds for topological reasons and is independent of any particular symmetry of the Lagrangian. In many of the theories studied, the kinks have been shown to be (classical analogs of) fermions (Williams and Zvengrowski, 1977). Conditions (i) and (ii) taken together then furnish a theoretical explanation of the experimentally observed conservation of fermion number.

The above analysis rests on the assumption that space or space–time is continuous. This assumption is generally accepted for large-scale phenomena, but Wheeler (1968) has pointed out that for general relativistic theories of phenomena on a scale of the order of the Planck length, 10^{-33} cm, quantum fluctuations in the curvature of space–time may be sufficient to produce alterations in topology. These violent changes in topology could be expected to affect the kink number. Suppose, for example, a space–time manifold that is $S^3 \times R^1$ changes to one that is $S^1 \times S^2 \times R^1$. Consider all type-(0,2) Lorentz metric tensor fields **g** on a space–time manifold $\mathfrak{M}^4$. Let Z denote the group of integers and Z_2 denote the group of integers modulo 2. When $\mathfrak{M}^4 = S^3 \times R^1$ the set of homotopy classes of **g** is isomorphic to Z, but when $\mathfrak{M}^4 = S^1 \times S^2 \times R^1$ it is isomorphic to $Z \oplus Z_2$ (Shastri, Williams, and Zvengrowski, 1980). Thus even the classifying scheme for the kinks is different in the two cases.

At a scale of 10^{-13} cm and below new particles can be created and it is known that certain quantum numbers (strangeness, parity, total isospin,...) are not always conserved. Thus the nonconservation of any kink labels that correspond to such quantum numbers would be an asset rather than a weakness of kink theory. However, quantities such as fermion number, baryon number, lepton number, and electric charge are known to be conserved under all circumstances and it would be most embarrassing for kink theory if kink labels corresponding to such quantities were not conserved at all times.

In this paper we shall investigate the problem of conservation/nonconservation of kinks for mappings between manifolds X and Y when the topology of the domain X is allowed to vary so that X can be replaced by a new manifold $\tilde{X}$ cobordant though not necessarily homeomorphic or homotopically equivalent to X. When X and Y have the same dimension it will be shown that the *degree* of mappings $\varphi: X \to Y$ is always conserved. This has an important consequence for general relativity, namely, that the number of fermions (i.e., kinks of half-odd-integer spin) is conserved.

First, we shall quote some standard results from bordism theory, many of which can be found in the book by Conner and Floyd (1964).

2. BORDISM, COBORDISM, AND HOMOLOGY

Let $X^n, \tilde{X}^n, Y^m$ be C^∞ compact oriented manifolds without boundary and of dimensions n, n, and m, respectively. Let

$$\begin{array}{ccc} X^n & \xrightarrow{f} & \\ & & Y^m \\ \tilde{X}^n & \xrightarrow{g} & \end{array}$$

be two C^∞ mappings. The C^∞ pairs (X^n,f) and $(\tilde{X}^n,g)$ are said to be *bordant* if and only if there exists a C^∞ pair (W^{n+1},F) where W^{n+1} is an $(n+1)$-manifold with boundary $\partial W^{n+1}=X^n \dot{\cup} (-\tilde{X}^n)$ and F is a mapping such that $F|X^n=f, F|\tilde{X}^n=g$. The dot denotes disjoint union, and $-\tilde{X}^n$ denotes the manifold $\tilde{X}^n$ with reversed orientation. If the above conditions hold for the special case in which Y^m is a single point then we say that the manifolds X^n and $\tilde{X}^n$ are *cobordant*. Clearly bordancy is a stronger condition than cobordancy since, for any Y^m, (X^n,f) being bordant to $(\tilde{X}^n,g)$ will imply that X^n will be cobordant to $\tilde{X}^n$.

A simple example of cobordancy is obtained by choosing $X^1=S^1$ and choosing $\tilde{X}^1=S^1 \dot{\cup} S^1$ (Figure 1). The manifold $\tilde{X}^1$ is not connected. The higher-dimensional generalizations of such "trouser worlds" are ruled out in general relativity (Kundt, 1967), and such situations will not be considered in this paper. Another example is obtained by choosing $X^2=S^2$ and $\tilde{X}^2=S^1\times S^1$. The manifold W^3 can be chosen to be the closed unit 3-disk (i.e., closed unit ball) with a hole shaped like a solid open torus cut out from the interior (Figure 2). Alternatively, W^3 can be chosen as a solid closed torus from the interior of which an open unit 3-disk has been removed (Figure 3).

We shall show below that bordism is an equivalence relation (i.e., reflexive, symmetric, and transitive). The equivalence class of the C^∞ pair (X^n,f) will be denoted by $[X^n,f]$. The set of these classes for a fixed Y^m is called the nth *bordism* homology group of Y^m and is denoted by $\Omega_n(Y^m)$. This set forms an Abelian group under the operation induced from disjoint union. The identity element is the class of any C^∞ sphere pair (S^n,f) where S^n denotes the standard unit n-sphere in R^{n+1}. The additive inverse of an element $[X^n,f]$ is $[-X^n,f]$. It only remains to prove transitivity, and this can be done by making use of the collaring lemma (Conner and Floyd,

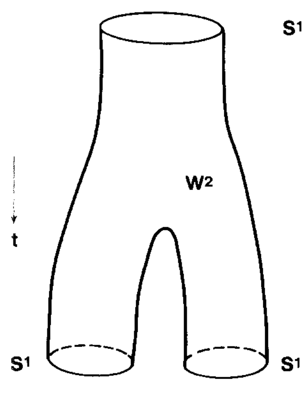

Fig. 1.

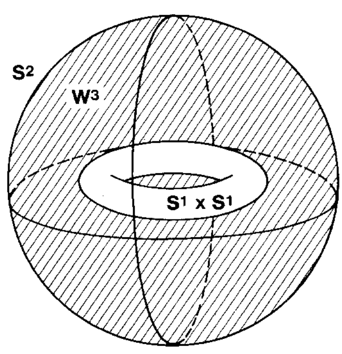

Fig. 2.

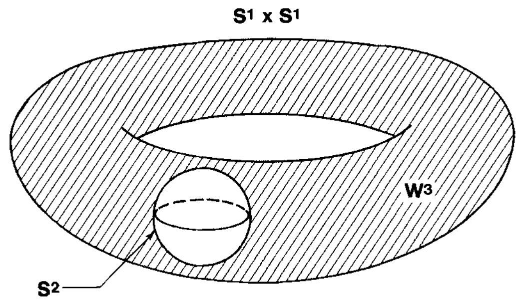

Fig. 3.

1964, p. 7), which states: For any compact, boundaryless, connected C^{∞} manifold X^n there is an open set U containing ∂X^n and a diffeomorphism φ of U onto $\partial X^n \times [0,1]$ with $\varphi(x)=(x,0)$ for all $x \in \partial X^n$. Hence if X^n is cobordant to $\tilde{X}^n$ via W_1^{n+1} and $\tilde{X}^n$ is cobordant to $\tilde{\tilde{X}}^n$ via W_2^{n+1}, the collaring lemma implies the existence of a diffeomorphism of an open neighborhood of $\tilde{X}^n$ in W_1^{n+1} to one of $\tilde{X}^n$ in W_2^{n+1}. Thus transitivity follows.

Suppose that Y^m is a single point that we denote by an asterisk. The group $\Omega_n(Y^m)$ becomes $\Omega_n(*)$, which we shall abbreviate to Ω_n. The group Ω_n is called the nth oriented *cobordism* group of Thom (or the oriented cobordism group of n-manifolds). It is a well-known fact that for $0<n\leqslant 3$, any two orientable n-manifolds are cobordant. Hence $\Omega_1 \approx \Omega_2 \approx \Omega_3 \approx 0$, where 0 denotes the group identity. In the case of 0-manifolds (i.e., points) it is clear that a manifold consisting of p points is cobordant with a manifold consisting of q points if and only if $p=q$. It follows that $\Omega_0 \approx Z$.

We shall now discuss the relationship between bordism groups and homology groups. Let Y be any manifold and let $H_n(Y;Z)$ denote the (singular) homology group of Y with integer coefficients. Let us recall how $H_n(Y;Z)$ is defined. Let Δ^n denote the standard n-simplex and call any continuous mapping $f:\Delta^n \to Y$ a "singular n-simplex." Consider the free Abelian group on singular n-simplices, i.e., the group of finite integral sums called "singular n chains," $\Sigma_i n_i f_i$, $n_i \in Z$. The operation of "taking the boundary" is essentially given by taking $\partial f = f|\partial\Delta^n$ and extending linearly to obtain the definition of $\partial(\Sigma_i n_i f_i)$ (Eilenberg and Steenrod, 1952). Of course, the definition is such that $\partial\partial=0$. An "n-cycle" is defined to be a singular n chain with vanishing boundary, i.e., $\partial(\Sigma_i n_i f_i)=0$. One obtains the homology equivalence relation on the set of n-cycles of Y by defining two n-cycles to be "homologous" if and only if they constitute the boundary of some singular $(n+1)$-chain. The set of homology equivalence classes of n-cycles of Y is denoted by $H_n(Y;Z)$. From this discussion it is clear that the bordism relation of C^∞ pairs is, strictly speaking, a special kind of singular homology. From this observation it follows that there is a natural homomorphism

$$\mu:\Omega_n(Y^m)\to H_n(Y^m;Z)$$

obtained by defining $\mu([X^n,f])=f_*(\sigma_n^X)$ where $\sigma_n^X\in H_n(X^n;Z)$ denotes the orientation class of X^n, and where

$$f_*:H_n(X^n;Z)\to H_n(Y^m;Z)$$

is the usual induced homomorphism on homology.

René Thom was the first to give an example of a manifold Y^m and an integer n such that μ is not an epimorphism (i.e., so that there exists a singular integral homology class of Y^m which is not the image of any bordism class under μ). The example necessarily involves odd torsion groups (i.e., finite groups in which every element apart from the identity has odd order) in the homology of Y^m because spectral sequence arguments due to Atiyah show that if there is no odd torsion in the homology of Y^m then μ is an epimorphism. In fact, in the case of no odd torsion there is a spectral sequence isomorphism (Conner and Floyd, 1964, p. 42)

$$\Omega_n(Y^m)\approx\sum_{m+p=n}H_m(Y^m;\Omega_p)$$

where Ω_p denotes the oriented cobordism group of Thom. These cobordism groups have been computed by Thom and Wall (Conner and Floyd, 1964).

We shall now consider the special case where $m=n$. Let $f:X^n\to Y^n$. If X^n,Y^n are orientable n-manifolds then $H_n(X^n;Z)\approx H_n(Y^n;Z)\approx Z$ and $f_*:H_n(X^n;Z)\to H_n(Y^n;Z)$ is simply multiplication by some integer N. The integer N is called the *degree* of f, $\deg f$ (Spanier, 1966). We wish to establish the important fact that the degree of the map f of a C^∞ pair (X^n,f) is preserved under the bordism equivalence relation.

Proposition. If (X^n,f) and $(\tilde{X}^n,g)$ are bordant

$$\begin{array}{ccc} X^n & \searrow^{f} & \\ & & Y^n \\ \tilde{X}^n & \nearrow_{g} & \end{array}$$

then $\deg f=\deg g$.

Proof. Consider

$$f_*(\sigma_n^X) = \mu([X^n, f]) = \mu([\tilde{X}^n, g]) = g_*(\sigma_n^{\tilde{X}})$$

If σ_n represents the orientation class in $H_n(Y^n; Z)$ then $f_*(\sigma_n^X) = \deg f \cdot \sigma_n$ and $g_*(\sigma_n^{\tilde{X}}) = \deg g \cdot \sigma_n$ and consequently $\deg f = \deg g$. ∎

This is particularly interesting because the manifolds X^n and $\tilde{X}^n$ need not be even homotopically equivalent when they are cobordant. The above proposition and the fact that all 3-manifolds are cobordant leads to the following corollary for the $n=3$ case.

Corollary. (X^3, f) and $(\tilde{X}^3, g)$ are bordant,

$$\begin{array}{ccc} X^3 & \overset{f}{\searrow} & \\ & & Y^3 \\ \tilde{X}^3 & \underset{g}{\nearrow} & \end{array}$$

if and only if $\deg f = \deg g$.

Proof. The "if" part follows from the spectral sequence isomorphism above. [The fact that $\Omega_0 \approx Z$ and $\Omega_1 \approx \Omega_2 \approx \Omega_3 \approx 0$ implies $\Omega_3(Y^3) \approx H_3(Y^3; Z) \approx Z$.] ∎

The important consequence of the above proposition is that kink labels associated with the degree of mapping will be conserved under topology change, although other kink labels may change. The relevance of this to general relativity will be discussed in the following section.

3. GENERAL RELATIVITY

Let the space–time manifold $\mathfrak{M}^4$ be a bundle space whose fiber is R^1 and whose base is any compact, boundaryless, connected, orientable 3-manifold, M. The compactness assumption for M is an artifice rather than a profound assumption about the large-scale topology of the universe. A topological structure of the type we have in mind is intended to be a model of an elementary particle, and hence the topologically interesting region of 3-space is limited in extent and can be contained within some 2-sphere. This is the region that we wish to study. It makes no difference to the topological analysis if the 2-sphere boundary is identified to a point. This will then produce a compact, boundaryless 3-manifold which we will call M.

The application of cobordism theory to general relativity has been discussed by many authors. In particular, Yodzis (1972, 1973) has studied so-called "Lorentz cobordism" and has shown it to be consistent with causality, geodesic completeness, and finite, positive energy density. A Lorentz cobordism requires an everywhere nonzero vector field to be placed on M. Since M is a 3-manifold, this is always possible. Thus throughout this section, for "(co)bordism" we could equally well read "Lorentz (co)bordism."

The kink labels of general relativity are the labels for the homotopy classes of type (0,2) Lorentz metric tensors g. The set of these homotopy classes is known to be isomorphic to $[M, RP^3]$, where RP^3 denotes real three-dimensional projective space. Two results due to Shastri, Williams, and Zvengrowski (1980) are relevant. The first of these states that $[M, RP^3]$ is always the direct sum of Z and a (possibly zero) number of Z_2 terms.

$$[M, RP^3] \approx Z \oplus Z_2 \oplus \cdots \oplus Z_2$$

The second result states that metrics corresponding to maps $\varphi: M \to RP^3$ which are of degree 2 will have half-odd-integer spin and so represent fermions—in the sense that it is possible to define double-valued spinor wave functionals $\Psi(\varphi)$ on the sectors of mapping space containing such maps. Indeed, a mapping of degree $2N$ will contain N fermions. [One may ask what happens if the degree is not an even number. For manifolds such as S^3 or $S^1 \times S^2$, all maps have even degree. However, for $M = RP^3$ maps of degree 1 can occur. The objects described by such odd-degree maps will represent kinks of a certain type, although their physical interpretation is unclear. They will not be considered in this paper, although further details are given by Shastri, Williams and Zvengrowski (1980).]

Given a C^∞ pair (M, φ) let us suppose that a change in topology leads to a new C^∞ pair $(\tilde{M}, \tilde{\varphi})$ that is bordant to the original (M, φ). Clearly, the labels associated with the Z_2 terms in $[M, RP^3]$ will not be conserved in general. However, the degree of mapping is conserved so that deg $\varphi = 2N$ implies deg $\tilde{\varphi} = 2N$. Thus the number of fermions present cannot change.

In the next section we treat a specific example where $M = S^3$ and $\tilde{M} = S^1 \times S^2$. The mapping $\varphi: S^3 \to RP^3$ is chosen to be the usual double covering of RP^3 by S^3. It is a map of degree 2 and represents one fermion (and, at the same time, one kink).

4. AN EXAMPLE

We shall consider $(\mathfrak{M}^4, \mathbf{g})$ for the case in which $\mathfrak{M}^4$ undergoes a change in topology from $S^3 \times R^1$ to $S^1 \times S^2 \times R^1$. As mentioned before, we need to consider mappings from some 3-manifold M into RP^3. In the notation of Section 2, let $X^3 = S^3$, $\tilde{X}^3 = S^1 \times S^2$ and $Y^3 = RP^3$. Mappings $f: S^3 \to RP^3$ are classified by a single integer $N \in Z$ and mappings $g: S^1 \times S^2 \to RP^3$ are classified by a pair of integers $(N,k) \in Z \oplus Z_2$. For both the X^3 case and the $\tilde{X}^3$ case, the integer N is usually taken to be *half* the degree of the map. Let $\kappa: S^3 \to RP^3$ be the usual double covering map. Note that deg $\kappa = 2$, and $N = 1$. Points in S^3, S^2 and $S^1 \times S^2$ are labeled as follows:

$$(\phi_1, \phi_2, \phi_3, \phi_4) = (\boldsymbol{\phi}, \phi_4) \in S^3, \qquad \sum_{\alpha=1}^{4} \phi_\alpha^2 = 1$$

$$(\mu_1, \mu_2, \mu_3) = \boldsymbol{\mu} \in S^2, \qquad \sum_{i=1}^{3} \mu_i^2 = 1$$

$$(\beta, \boldsymbol{\mu}) \in S^1 \times S^2, \qquad 0 \leqslant \beta \leqslant 2\pi$$

Define $\psi: [0, 2\pi] \times S^2 \to S^3$ by

$$\psi(\beta, \boldsymbol{\mu}) = \left(\boldsymbol{\mu} \sin\tfrac{1}{2}\beta, \cos\tfrac{1}{2}\beta\right)$$

Define the quotient map (i.e., identification map) $q: [0, 2\pi] \times S^2 \to S^1 \times S^2$ by

$$q(\beta, \boldsymbol{\mu}) = (\beta, \boldsymbol{\mu}) \qquad \text{for } \beta \neq 0, 2\pi$$

and

$$q(0, \boldsymbol{\mu}) = q(2\pi, \boldsymbol{\mu}) = (*, \boldsymbol{\mu})$$

where * is the base point in S^1. The following diagram commutes:

$$\begin{array}{ccc} [0,2\pi] \times S^2 & \xrightarrow{\ \psi\ } & S^3 \\ \big\downarrow q & & \big\downarrow \kappa \\ S^1 \times S^2 & \overset{\tilde{\psi}}{\dashrightarrow} & RP^3 \end{array}$$

$\tilde{\psi}$ is the unique induced map which agrees identically with ψ away from $(*, S^2)$ and is the antipodal map on $(*, S^2)$ [i.e., $\tilde{\psi}(-x) = -\tilde{\psi}(x)$]. Note the importance of the $\frac{1}{2}\beta$ factor in the formula for ψ. Because of this, we obtain deg $\tilde{\psi} = 2$. Note that $\tilde{\psi}$ is a map of type $(1, 1) \in Z \oplus Z_2$.

We now give an explicit construction of a suitable manifold W^4 and mapping $F: W^4 \to RP^3$ which is a bordism between $\tilde{\psi}$ and κ. The mapping

ψ extends to

$$\hat{\psi}:[0,2\pi]\times(D^3-C)\to S^3$$

where C is a small open 3-disk about the origin of the closed unit 3-disk D^3, via

$$\hat{\psi}(\beta,\boldsymbol{\nu})=\left(\frac{\boldsymbol{\nu}}{\|\boldsymbol{\nu}\|}\sin\tfrac{1}{2}\beta,\cos\tfrac{1}{2}\beta\right)$$

where $0<\|\boldsymbol{\nu}\|^2=\sum_{i=1}^{3}\nu_i^2\leqslant 1$. Furthermore, the above commutative diagram extends to

$$\begin{array}{ccc}
[0,2\pi]\times(D^3-C) & \xrightarrow{\hat{\psi}} & S^3 \\
\Big\downarrow Q & & \Big\downarrow \kappa \\
S^1\times(D^3-C) & \overset{\tilde{\psi}}{\dashrightarrow} & RP^3
\end{array}$$

by using the diffeomorphisms

$$[0,2\pi]\times(D^3-C)\approx[0,2\pi]\times S^2\times[1,l]$$

and

$$S^1\times(D^3-C)\approx S^1\times S^2\times[1,l]$$

where $l=\|\boldsymbol{\nu}\|$ is a radial coordinate, $l\neq 0$. The quotient map Q is defined by

$$Q(\beta,\boldsymbol{\nu})=\left(q\left(\beta,\frac{\boldsymbol{\nu}}{\|\boldsymbol{\nu}\|}\right),\|\boldsymbol{\nu}\|\right)$$

Thus Q preserves the radial coordinate l.

We now show the existence of a continuous extension of $\tilde{\psi}$ to $S^1\times D^3$. First of all, Q extends to $[0,2\pi]\times D^3$ by

$$Q(\beta,\boldsymbol{\nu})=\begin{cases}\left(q\left(\beta,\frac{\boldsymbol{\nu}}{\|\boldsymbol{\nu}\|}\right),\|\boldsymbol{\nu}\|\right), & \boldsymbol{\nu}\neq\mathbf{0}\\ (\beta(\mathrm{mod}\,2\pi),\mathbf{0}), & \boldsymbol{\nu}=\mathbf{0}.\end{cases}$$

This works because the pair

$$\left(\frac{\boldsymbol{\nu}}{\|\boldsymbol{\nu}\|},\|\boldsymbol{\nu}\|\right)\in S^2\times[1,l]$$

is identified with $\nu \in D^3 - C$ and

$$q\left(\beta, \frac{\nu}{\|\nu\|}\right) = \left(\beta(\operatorname{mod} 2\pi), \frac{\nu}{\|\nu\|}\right)$$

Secondly, $\hat{\psi}$ is clearly not onto for it misses (0,0,0,1) and it follows that there exists an open 3-disk around (0,0,0,1) which does not intersect the image of $\hat{\psi}$. Hence the image of $\hat{\psi}$ lies in a closed 3-disk I^3. Since I^3 is homeomorphic to the standard 3-cube we may apply the Tietze extension theorem (Dugundji, 1966) to obtain a continuous map of the whole of $[0,2\pi] \times D^3$. With a mild abuse of language, we shall denote this extension by $\hat{\psi}$ also. We then have the diagram

$$\begin{array}{ccc} [0,2\pi] \times D^3 & \xrightarrow{\hat{\psi}} & S^3 \\ \downarrow Q & & \downarrow \kappa \\ S^1 \times D^3 & \overset{\tilde{\hat{\psi}}}{\dashrightarrow} & RP^3 \end{array}$$

where $\tilde{\hat{\psi}}$ is the unique map which makes the diagram commute and extends the original $\tilde{\psi}$ whose domain was $S^1 \times (D^3 - C)$. Note that $\tilde{\hat{\psi}}|S^1 \times C$ is just the Tietze map followed by κ on $(0,2\pi) \times C$.

If we consider a small arc A through the base point * in S^1 and let B denote an open 4-disk with ∂B in the complement of $A \times C$, then we can define

$$\check{\psi} = \tilde{\hat{\psi}} | \partial B = S^3$$

which is then given by the formula for $\hat{\psi}$, above, away from the set $\{(\beta, \nu) | \nu \in C\}$. We take $W^4 = S^1 \times D^3 - B$ and define the bordism map F to be $\hat{\psi}| W^4$. This map is not necessarily C^∞ on $S^1 \times \bar{C}$. However, F may be approximated in a standard way by a C^∞ map homotopic to the original and ε-close to it (relative to any convenient Riemannian metric on RP^3) and which agrees with the original F wherever it is C^∞ (Steenrod, 1951, pp. 25–28).

Now $\check{\psi}$ has degree two on S^3, since it is bordant to a map of degree two. By the corollary proved above, equal degrees imply bordism, so that $\check{\psi}$ and κ are bordant. We attach this bordism to (W,F) along $(S^3, \check{\psi})$ and the transitivity of the bordism relation yields the desired bordism of $(S^1 \times S^2, \tilde{\psi})$ and (S^3, κ).

5. SUMMARY AND CONCLUSIONS

We have studied the case of field theories in which the domain manifold of the field variables is allowed to change to a different, although cobordant, manifold. It is pointed out that the degree of the mapping defined by the field variables will always be conserved. In general relativity, it is known that fermions can arise as topological half-odd-integer spin structures related to the homotopy class of the metric. This paper has shown that, in spite of the violent fluctuations of topology in the "space–time foam" of the manifold, the number of such topological fermions is strictly conserved.

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